Convergence of the Point Vortex Method for the 2-D Euler Equations

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Abstract

We prove consistency, stability and convergence of the point vortex approximation to the 2-D incompressible Euler equations with smooth solutions. We first show that the discretization error is second-order accurate. Then we show that the method is stable in l_p norm. Consequently the method converges in l_p norm for all time. The convergence is also illustrated by a numerical experiment.

We prove consistency, stability, and convergence of the point vortex approximation to the incompressible Euler equations in two dimensions. The vortex method for this problem (see e.g. Chorin [5]) has two parameters, the grid length, h, and the smoothing length (or "blob size"), δ. In computational practice people usually take $\delta \sim h$ or $\delta = 0$ (see [13]) but the convergence theorem of Hald [10] and subsequent work needed the hypothesis that $\delta \gg h$, leaving these cases of computational practice in doubt. In this paper we prove convergence with $\delta = 0$ under hypotheses that are otherwise similar to those of Hald. This convergence is also illustrated by a numerical experiment. Of course, we can not claim that the point vortex method is better than the vortex blob method or vice versa; our purpose is to show that the point vortex method works in principle. In future work [9], we shall treat the case $\delta \neq 0$ but δ not much larger than h. We shall also treat some of the proposed three-dimensional vortex methods; see [11].

The proof follows that of Hald with an improvement (l_p instead of l_2 norms) introduced by Beale and Majda [3]. We show that the method is formally second-order accurate by showing that certain sums are second-order accurate approximations to the corresponding Biot Savart integrals. Actually, we get an asymptotic expansion for the truncation error that would allow us to use Strang's device in [15] and prove convergence for low-order accurate methods in high dimensions. That is deferred to a later paper. We then prove l_p stability for $1 < p < \infty$ by using L_p bounds for a related continuous singular integral operator, much as was done by Hald and by Beale and Majda. Except for this L_p bound, our proof is elementary and selfcontained.

In vorticity formulation, the two-dimensional inviscid incompressible Euler equations are (see [10])

$$(1) \qquad \omega_t + (u \cdot \nabla)\omega = 0,$$

$$(2) \qquad u(x,t) = \int K(x-y)\omega(y,t)\,dy,$$

$$(3) \qquad K(x) = \frac{1}{2\pi|x|^2} \cdot (-x_2, x_1).$$

It is known (see e.g. [16]) that for the initial vorticity $\omega(x,0) = \omega^0(x)$ which is smooth and localized ($\omega^0 \in S$, the Schwartz class; see [8]) a solution of (1), (2) exists for all times ($\omega(\cdot, t) \in S$ for all $t > 0$). The Lagrangian (or "flow map") coordinate, ξ, is defined implicitly by the particle trajectories, $y(\xi, t)$, which satisfy

$$(4) \qquad \frac{d}{dt}y(\xi,t) = u(y(\xi,t),t), \qquad y(\xi,0) = \xi.$$

If $\omega^0 \in S$, then this change of variables is smooth. Since the flow is incompressible, $\det(\partial y/\partial\xi) \equiv 1$ and therefore the mapping $\xi \to y$ has a smooth inverse $\xi = \xi(y,t)$. The point vortex method approximation is to replace (2) and (1) by

$$(5) \qquad v_j(t) = h^2 \sum_{k \neq j} K(x_j(t) - x_k(t))\omega_k,$$

and

$$(6) \qquad \dot{x}_j(t) = v_j(t),$$

where $x_j(0) = h \cdot j$, $j = (j_1, j_2) \in \mathbf{Z}^2$, $h \in \mathbf{R}$ is the grid length, and $\omega_j = \omega^0(x_j(0))$. A consequence of localized vorticity is that particles far away move very little. In particular, for any T we have

$$\lim_{r \to \infty} \sup_{\substack{|\xi| \geq r, \\ t \leq T}} |y(\xi,t) - \xi| = 0.$$

From (1) it follows that

$$\omega(y(\xi,t),t) = \omega(\xi,0) = \omega^0(\xi).$$

The exact particle trajectories $y_j(t)$ are the Lagrangian images of the original grid: $y_j(t) = y(\xi_j, t)$. We write $\xi_j = h \cdot j$ so that $\omega(y_j(t), t) = \omega^0(\xi_j) = \omega_j$. The

discrete l_p norm is defined by

$$\|f\|^p = h^2 \sum_j |f_j|^p.$$

Recall that $v(t)$ is the approximate velocity field defined by (5). We also write $u_j = u(y_j(t), t)$, the exact solution. Our result is

THEOREM A. *For all T and $4 < p < \infty$, there is a $C(T, p)$ such that*

$$(7) \qquad \|x(t) - y(t)\| \leqq C(T, p) h^2,$$

$$(8) \qquad \|v(t) - u(t)\| \leqq C(T, p) h^2.$$

THEOREM B. *Suppose $\omega^0 \in S$, $4 < p < \infty$, and $T > 0$ are given, and that*

$$(9) \qquad \|x(t) - y(t)\| \leqq h^{2-1/4} \quad \text{for} \quad 0 \leqq t \leqq T.$$

Then there exists an $A(T, p)$ such that

$$(10) \qquad \|v(t) - u(t)\| \leqq A(T, p)\|x(t) - y(t)\| + A(T, p) h^2,$$

for $0 \leqq t \leqq T$, where $A(T, p)$ is a nondecreasing function in T.

Remarks. 1. The assumption $\omega^0 \in S$ can be replaced by a finite degree of differentiability and decay.

2. The $h^{2-1/4}$ in (9) can be replaced by $\|x(t) - y(t)\|_{l_\infty} \leqq \phi(h)h$ for any positive function $\phi(h)$ with $\phi(h) \to 0$ as $h \to 0$.

3. By a longer consistency argument that uses the asymptotic error expansion (17) in the manner of Strang [15], the restriction $4 < p < \infty$ can be replaced by $1 < p < \infty$.

4. Our Lemma 1 to follow is in the spirit of the remark of Beale [2] on the consistency of a 3-D vortex blob method.

5. The techniques in this paper also apply to the deterministic vortex method of Cottet and Mas-Gallic [6] for the incompressible Navier-Stokes equations.

6. Our result does not apply to the ill-posed vortex sheet problem where analyticity seems to be necessary for convergence; see [4]. .

We choose to present a proof that does not depend on smoothing the kernel K as in the blob method. We mention, however, that due to the incompressibility and smoothness of the flow map $y(\xi)$, the point vortex method can be identified with the blob method with blobs of size $O(h)$; see [11]. Consequently, the blob techniques can be more or less applied directly resulting in a shorter but less selfcontained proof. We do not present this method to emphasize that no additional smoothing is necessary.

The statement of direct practical interest is Theorem A, but it is more convenient to prove Theorem B. Let us indicate how Theorem A follows from Theorem B.

Proof of Theorem A from Theorem B: For a given h and p, define T^* by

$$T^* = \inf \left\{ t : \|x(t) - y(t)\| \geq h^{2-1/4} \right\}.$$

It suffices to show that if $h \leq \frac{1}{16} \exp\{-4A(T, p) \cdot T\}$, then $T^* > T$. For in that case ($T^* > T$), we have $\|x(t) - y(t)\| \leq h^{2-1/4}$ for all $t \leq T$. It follows from (10) that, for $t \leq T$,

$$\frac{d}{dt} \|x(t) - y(t)\| \leq \|v(t) - u(t)\| \leq A(T, p)\|x(t) - y(t)\| + A(T, p)h^2,$$

which, via Gronwall's inequality, proves (7) with $C = e^{A \cdot T}$. We get (8) with $C = Ae^{A \cdot T}$ from (7) and (10). Now, if $T^* \leq T$, we have, for all $t \leq T^*$,

$$\frac{d}{dt} \|x(t) - y(t)\| \leq \|v(t) - u(t)\| \leq A(T^*, p)\|x(t) - y(t)\| + A(T^*, p)h^2,$$

by (10), which implies that $\|x(t) - y(t)\| \leq \exp\{A(T^*, p)T^*\}h^2$ for $t \leq T^*$ by Gronwall's inequality, But $A(T^*, p)$ is a nondecreasing function of T^*. By the choice of h, we conclude that

$$\|x(t) - y(t)\| \leq \exp\{A(T, p)T\}h^2 \leq \tfrac{1}{2}h^{2-1/4},$$

for all $t \leq T^*$, which contradicts the definition of T^*.

Proof of Theorem B: First we show that the y_j are close to satisfying (5) and (6). More precisely, define the residual, or "truncation error", ρ, by

$$(11) \qquad \rho_j(t) = u\big(y_j(t), t\big) - h^2 \sum_{k \neq j} K\big(y_j(t) - y_k(t)\big)\omega_k.$$

We claim:

LEMMA 1 (Consistency).   *There is a constant, $D(T)$, such that $\|\rho(t)\| \leq D(T)h^2$ for all $t \leq T$.*

Proof of Lemma 1: In (2) we change from y to ξ as variable of integration and set $x = y_j(t)$. Using incompressibility, we have

$$u\big(y_j(t), t\big) = \int K\big(y_j(t) - y(\xi, t)\big)\omega^0(\xi)\, d\xi.$$

Thus ρ is the error in trapezoid rule integration. Therefore the lemma would be routine but for the singularity of K. To handle this we split ρ into its near field part, σ, and its far field part, τ. For this, we use a smooth cut-off function, $f(s)$, of $s \in \mathbf{R}$, satisfying $f(s) \equiv 1$ if $s \leq 1$ and $f(s) \equiv 0$ if $s \geq 2$. Let $g(s) = 1 - f(s)$ be the complement of f. We split up ρ_j into $\sigma_j + \tau_j$, where $0 < q < 1$ and

$$
(12) \quad
\begin{aligned}
\sigma_j = & \int K\big(y_j - y(\xi)\big)\omega^0(\xi)f\big(h^{-q}|\xi_j - \xi|\big)\, d\xi \\
& - h^2 \sum_{k \neq j} K(y_j - y_k)\omega^0(\xi_k)f\big(h^{-q}|\xi_j - \xi_k|\big)
\end{aligned}
$$

and

$$
(13) \quad
\begin{aligned}
\tau_j = & \int K\big(y_j - y(\xi)\big)\omega^0(\xi)g\big(h^{-q}|\xi_j - \xi|\big)\, d\xi \\
& - h^2 \sum_{k \neq j} K(y_j - y_k)\omega^0(\xi_k)g\big(h^{-q}|\xi_j - \xi_k|\big).
\end{aligned}
$$

For notational convenience, we temporarily take $j = (0,0)$ so that $\xi_j = (0,0)$ and we write y' for $y'(0)$, the derivative of y with respect to ξ, etc.

To estimate the near field, we must characterize the near field behavior of $K(y(0) - y(\xi))\omega^0(\xi)$ for small ξ.

LEMMA 2. *For any N, we have*

$$
(14) \quad K(-y(\xi))\omega^0(\xi) = \frac{b \cdot \xi}{|y'\xi|^2} + m_0(\xi) + m_1(\xi) + \cdots + O(|\xi|^N),
$$

where b is a 2×2 matrix and m_n is homogeneous of degree n in ξ: $m_n(\alpha\xi) = \alpha^n m_n(\xi)$ for all $\alpha \in \mathbf{R}$.

The denominator does not vanish since $\det(y') = 1$. The terms b and m_n depend on t, but they as well as the error bound implied in (14) are uniform in $t \leq T$. Since the m_n are actually rational functions of ξ, α need not be positive. In particular, $m_n(\xi)$ is an even or odd function of ξ depending on whether n is even or odd.

Proof of Lemma 2: Note that since $y(\xi, t)$ and $\omega^0(\xi)$ are C^∞ functions of ξ, we have the expansions

$$
(15) \qquad y(\xi) = y(0) + y'\xi + y''(\xi, \xi) + \cdots + O(|\xi|^N)
$$

and

$$
(16) \qquad \omega^0(\xi) = \omega^0(0) + \omega^{0\prime}\xi + \omega^{0\prime\prime}(\xi, \xi) + \cdots + O(|\xi|^N)
$$

for any N. The numerator of K has a simple expansion, so we need only study the denominator

$$\frac{1}{|y(\xi)^2|} = \frac{1}{y_1^2 + y_2^2}.$$

But, using (15), we get

$$|y|^2 = |y'\xi|^2 \cdot \left(1 + \frac{2(y_1' \cdot \xi) \cdot y_1''(\xi, \xi) + 2(y_2' \cdot \xi) \cdot y_2''(\xi, \xi)}{|y'\xi|^2} + \cdots\right),$$

where the successive terms have the orders and smoothness claimed. Once we have an expansion for K using (15), we can multiply it by (16) to get the desired expansion for $K\omega$, which proves the lemma.

Consider the integral term in (12). Since $f(|\xi|)$ is even, we have

$$\int \frac{b \cdot \xi}{|y'\xi|^2} f(h^{-q}|\xi|) \, d\xi = 0,$$

and

$$\int m_n(\xi) f(h^{-q}|\xi|) \, d\xi = 0 \quad \text{for} \quad n \text{ odd}.$$

Also, the substitution $\eta = h^{-q}\xi$ gives, for n even,

$$\int m_n(\xi) f(h^{-q}|\xi|) \, d\xi = a_n h^{(n+2)q} \quad \text{where} \quad a_n = \int m_n(\eta) f(|\eta|) \, d\eta.$$

A similar analysis holds for the discrete summation terms of (12). The odd order terms vanish. For the even terms we have a little trick. The sums may be written as

$$h^2 \sum_{k \neq 0} m_n(hk) f(h^{1-q}|k|) = h^{n+2} S_n(h),$$

where

$$S_n(h) = \sum_{k \neq 0} m_n(k) f(h^{1-q}|k|).$$

Now, since S_n is a sum with finitely many non-zero terms,

$$\frac{d}{dh} S_n(h) = (1-q) \sum_{k \neq 0} m_n(k) \nabla f(h^{1-q}|k|) \cdot h^{-q}|k|$$

$$= (1-q) h^{-(n+3)(1-q)-q} \cdot \sum_{k \neq 0} m_n(h^{1-q}k) \tilde{f}(h^{1-q}|k|)(h^{1-q})^2,$$

where $\tilde{f}(y) = \nabla_y f(|y|) \cdot |y|$. The sum is a trapezoid rule approximation to

$$\tilde{b}_n = \int m_n(y)\tilde{f}(y)\,dy.$$

The integrand is a C_0^∞ function of y since ∇f vanishes at the origin where m_n is singular and ∇f has compact support. It is well known (e.g. [1], [7], p. 300) that the trapezoid rule for such integrals has an infinite order of accuracy, that is

$$\left(h^{1-q}\right)^2 \sum_{k \neq 0} m_n(h^{1-q}k)\tilde{f}(h^{1-q}k) = \tilde{b}_n + O(h^N) \quad \text{for any} \quad N.$$

This shows that

$$\frac{d}{dh} S_n(h) = (1-q)h^{-(n+3)(1-q)-q}\tilde{b}_n + O(h^N).$$

Using this and the integral representation

$$S_n(h) = S_n(1) - \int_h^1 \frac{d}{d\bar{h}} S_n(\bar{h})\,d\bar{h}$$

we obtain

$$S_n(h) = c_n + b_n h^{-(n+2)(1-q)} + O(h^N) \quad \text{for any} \quad N,$$

where c_n is a constant and $b_n = -\tilde{b}_n/(n+2)$.

Putting all this together and returning to the general case $j \neq 0$ gives

$$\sigma_j = \left(c_2(y_j, t)h^2 + c_4(y_j, t)h^4 + \cdots\right)$$

$$+ \left(d_2(y_j, t)h^{2q} + d_4(y_j, t)h^{4q} + \cdots\right) + O(h^N) \quad \text{for any} \quad N.$$

The coefficients c_n and d_n are functions of derivatives of ω and y at the point y_j which decay rapidly at infinity if $\omega \in S$. Comparing the expansion corresponding to $\frac{1}{2} < q < 1$ with the expansion corresponding to $0 < q < \frac{1}{2}$, we conclude that $d_n = 0$ for n even. That is,

$$\sigma_j = c_2(y_j, t)h^2 + c_4(y_j, t)h^4 + \cdots + O(h^N) \quad \text{for any} \quad N.$$

The theory of numerical integration tells us that $\tau_j = O(h^N)$ for any N, since the integrand is without singularities (thanks to g) and decays at infinity (thanks to ω). Combining this with our estimate for σ_j, we conclude that

$$(17) \quad \rho_j(h) = c_2(\xi_j, t)h^2 + c_4(\xi_j, t)h^4 + \cdots + O(h^N) \quad \text{for any} \quad N.$$

Since the coefficients $c_n(y)$ are rapidly decaying as $y \to \infty$, summing over j shows that $\|\rho\|$ is $O(h^2)$; this proves Lemma 1.

We note that it is possible to obtain second-order accuracy by using the oddness of the most singular part of the kernel and using straightforward estimates involving the midpoint rule (see [11]).

We turn now to proving (10). We use the following estimate: For any mesh function f_j,

$$(18) \qquad h^2|f_j|^p \leq h^2 \sum_k |f_k|^p = \|f\|^p;$$

thus

$$|f_j| \leq h^{-2/p}\|f\|.$$

In particular, applying (18) to $f = x(t) - y(t)$ and using (9) gives us

$$(19) \qquad \left|x_j(t) - y_j(t)\right| \leq h^{2-1/4-2/p}.$$

We conclude that, for $p > 4$, the left side of (19), the error in any particular particle position, is small relative to h, the typical distance between particles.

Another useful fact is Young's inequality (e.g. [8], p. 13). If $A = \{a_{jk}\}$ is an $N \times N$ matrix, and

$$M_r = \max_j \sum_k |a_{jk}| \quad \text{and} \quad M_c = \max_k \sum_j |a_{jk}|,$$

(the subscripts r and c stand for row and column, respectively) then

$$\|A\| \leq \max(M_r, M_c).$$

Similarly, if $L(x, y)$ is an integral kernel with

$$M_r = \sup_x \int_y |L(x, y)| \, dy \quad \text{and} \quad M_c = \sup_y \int_x |L(x, y)| \, dx,$$

then

$$\|L\|_{L_p} \leq \max(M_r, M_c).$$

We are now ready to estimate $v - u$. Using (5) and (11), we have

$$v_j(t) - u_j(t) = h^2 \sum_{k \neq j} \left(K(x_j - x_k) - K(y_j - y_k) \right) \omega_k + \rho_j.$$

Using a Taylor expansion for each component of K, we can write this as

$$v_j(t) - u_j(t) = h^2 \sum_{k \neq j} \nabla K(y_j - y_k)\big((x_j - y_j) - (x_k - y_k)\big)\omega_k$$

$$+ h^2 \sum_{k \neq j} \nabla^2 K(z_j - z_k)\big[x_j - y_j - (x_k - y_k)\big]\omega_k + \rho_j,$$

where $\nabla^2 K(x)[y]$ stands for the matrix of second partials acting as a quadratic form on the argument that appears in square braces. From the mean value theorem, z_j is somewhere on the line segment connecting x_j to y_j, and z_k is on the segment connecting x_k to y_k. (This is an abuse of notation since z_j depends on k and vice versa.) It follows from (19) that

$$(20) \qquad\qquad\qquad z_j = y_j + o(h).$$

Decompose $v - u$ as a sum of $T_1 + T_2 + T_3 + \rho$, where

$$T_1 = h^2\left(\sum_{k \neq j} \nabla K(y_j - y_k)\omega_k\right)(x_j - y_j),$$

$$T_2 = h^2 \sum_{k \neq j} \nabla K(y_j - y_k)\omega_k(x_k - y_k),$$

$$T_3 = h^2 \sum_{k \neq j} \nabla^2 K(z_j - z_k)\big[x_j - y_j - (x_k - y_k)\big]\omega_k.$$

Lemma 1 gives an l_p bound for ρ. We shall derive l_p bounds for each of the T_i of the form $\|T_i\| \leq C\|x - y\|$ for $i = 1, 2, 3$.

Using the definition (3) of K and (20), we bound the second derivative term by

$$\left|\nabla^2 K(z_j - z_k)\right| \leq \frac{C}{|z_j - z_k|^3} \leq \frac{C}{\left|y_j - y_k + o(h)\right|^3}.$$

Since ξ is a smooth function of y, and since $|y_j - y_k| \geq c(t)|\xi_j - \xi_k| \geq c(t)h$ for $j \neq k$, the right side is bounded by $C(t)/(|\xi_j - \xi_k|^3)$. Estimating the sum by the corresponding integral gives

$$h^2 \sum_{\substack{k \\ k \neq j}} \left|\nabla^2 K(z_j - z_k)\omega_k\right| \leq C(t)\int_{|\xi - \xi_j| > ah}\frac{|\omega(\xi)|}{|\xi_j - \xi|^3}\,d\xi \leq C(t)h^{-1},$$

where $\alpha(t) > 0$. With this, we have, using (19) and Young's inequality,

$$\|T_3\| \leq A(p, T) h^{1-1/4-2/p} \|x(t) - y(t)\|.$$

To handle T_1 and T_2 we use Hald's trick of dividing space into Lagrangian boxes. These are defined by (here $y = (y_1, y_2)$ and $j = (j_1, j_2)$):

$$B_j(t) = \left\{ y(\xi, t) : \max(|\xi_1 - hj_1|, |\xi_2 - hj_2|) \leq \tfrac{1}{2}h \right\}$$

and

$$\overline{B}_j(t) = \left\{ y(\xi, t) : \max(|\xi_1 - hj_1|, |\xi_2 - hj_2|) \leq \tfrac{1}{4}h \right\}.$$

The smaller boxes, $\overline{B}_j$, seem necessary to keep points in one small box separated from points in neighboring boxes. Because of the volume preserving property of $\xi(y)$,

$$(21) \qquad \int_{B_j(t)} dy \equiv h^2 \quad \text{and} \quad \int_{\overline{B}_j(t)} dy \equiv \tfrac{1}{4}h^2.$$

We first use the decomposition into boxes to bound the sums in T_1:

$$d_j = h^2 \sum_{k \neq j} \nabla K(y_j - y_k) \omega_k,$$

which are approximations to

$$\int \nabla K(y_j - y(\xi)) \omega^0(\xi) \, d\xi.$$

Since T_1 is a diagonal matrix with diagonal entries d_j acting on $x_j - y_j$, the l_p bound on T_1 follows from the boundedness of the individual d_j. First, by the smoothness of ω and y, and the approximate oddness of K (as in Lemma 1), we have

$$\left| \int_{B_j(t)} \partial_{x_1} K(y_j - y) \omega(y, t) \, dy \right|$$

$$(22) \qquad = \left| \omega^0(\xi_j) \int_{B_j(t)} \partial_{y_1} K(y_j - y) \, dy \right| + O(h)$$

$$= \left| \omega^0(\xi_j) \int_{\partial B_j(t)} K(y_j - y) \, dy_1 \right| + O(h) = O(h).$$

Now, we integrate over each box (ignoring, as we may by (22), B_j) and sum over the boxes to get

$$\left| h^2 \sum_{\substack{k \\ k \neq j}} \nabla K(y_j - y_k)\omega_k^0 - \int \nabla K(y_j - y(\xi))\omega^0(\xi)\, d\xi \right|$$

$$\leq \left| h\int_{|\xi - \xi_j| \geq \alpha h} \frac{C}{|\xi - \xi_j|^3}\omega^0(\xi)\, d\xi \right| \leq C(t),$$

by the midpoint rule (see [7]), where we have used the fact that $|y(\xi_j) - y(\xi)| \geq \alpha(t)|\xi_j - \xi| \geq \alpha(t)h$ for $y(\xi) \notin B_j(t)$.

Thus, we have

$$|d_j| = \left| h^2 \sum_{k \neq j} \nabla K(y_j - y_k)\omega_k \right| \leq C(t).$$

This settles T_1 but leaves T_2. For this, we need an l_p bound for the matrix whose elements are $h^2\nabla K(y_j - y_k)\omega_k$. But multiplication by ω_k is bounded, so we only need:

LEMMA 3 (Stability). *The matrix M whose elements are*

$$m_{jk} = \begin{cases} h^2\nabla K(y_j - y_k) & \text{if } j \neq k, \\ 0 & \text{if } j = k, \end{cases}$$

has the l_p bound $\|M\| \leq C(t, p)$.

Proof of Lemma 3: Suppose f is any grid vector and $g = Mf$. We want to show that $\|g\| \leq C(t, p)\|f\|$. For this (following Hald), we introduce functions of y. First, $F(y)$ is defined in terms of f by $F(y) = 4f_j$ if $y \in \bar{B}_j$, and by $F(y) = 0$ if $y \notin \bigcup_j \bar{B}_j$. The factor 4 is such that, in view of (21), $\|f\|_{l_p} = \|F\|_{L_p}$. In the same way, $G(y)$ is defined in terms of g. Define a piecewise constant kernel

$$L_h(y, y') = \begin{cases} \nabla K(y_j - y_k) & \text{if } y \in B_j(t), \ y' \in B_k(t) \text{ and } j \neq k, \\ 0 & \text{otherwise,} \end{cases}$$

and the corresponding "exact" kernel

$$L(y, y') = \nabla K(y - y').$$

The characteristic function for $\bar{B}_j$ is

$$\bar{\chi}_j(y) = \begin{cases} 1 & \text{if } y \in \bar{B}_j, \\ 0 & \text{otherwise,} \end{cases}$$

and the characteristic function for $\bigcup_j \bar{B}_j$ is $\bar{\chi}(y) = \sum_j \bar{\chi}_j(y)$:

$$\bar{\chi}(y) = \begin{cases} 1 & \text{if } y \in \bigcup_j \bar{B}_j, \\ 0 & \text{otherwise.} \end{cases}$$

Schematically, the proof is as follows:

$$\|g\|_{l_p} = \|G\|_{L_p} = 4\|\bar{\chi} \cdot L_h \cdot \bar{\chi} F\|_{L_p} \overset{?}{\leq} 4C\|LF\|_{L_p} \overset{?}{\leq} 4C'\|F\|_{L_p} = 4C'\|f\|_{l_p},$$

where $LF(y) = \int L(y, y')F(y')\, dy'$. We must establish the two questioned inequalities. The second of them is the Calderon Zygmund inequality (e.g. [11], p. 29).

The remaining inequality is proved by showing that

$$\|\bar{\chi}(L_h - L)\bar{\chi}\|_{L_p} \leq C(t).$$

We divide the kernel

$$R(y, y') = \bar{\chi}(y)\big(L_h(y, y') - L(y, y')\big)\bar{\chi}(y')$$

into two parts: $R = R_1 + R_2$. First we take the case when y and y' are in the same small box:

$$R_1(y, y') = \begin{cases} R(y, y') & \text{if } y \in \bar{B}_j \text{ and } y' \in \bar{B}_j \text{ for some } j, \\ 0 & \text{otherwise} \end{cases}$$

If $R_1(y, y') \neq 0$, then $L_h(y, y') = 0$; thus

$$\|R_1 F\|_{l_p}^p \leq \sum_j \int_{\bar{B}_j} |L\bar{\chi}_j F|^p\, dy \leq \sum_j \|L(\bar{\chi}_j F)\|_{l_p}^p$$

$$\leq \|L\|_{l_p}^p \sum_j \|\bar{\chi}_j F\|_{l_p}^p \leq \|L\|_{l_p}^p \|F\|_{l_p}^p.$$

By the Calderon-Zygmund inequality, we get $\|R_1\|_{L_p} \leq B(p) < \infty$. Now we look at the part coming from different boxes:

$$R_2(y, y') = \bar{\chi}(y)\big(\nabla K(y_j - y_k) - \nabla K(y - y')\big)\bar{\chi}(y')$$

$$\text{if} \quad y \in \bar{B}_j \quad \text{and} \quad y' \in \bar{B}_k \quad \text{and} \quad j \neq k.$$

For R_2, we use Young's inequality. The usual mean value argument gives

$$|\nabla K(y_j - y_k) - \nabla K(y - y')| \leq \big(|y_j - y| + |y_k - y'|\big) \cdot |\nabla^2 K(z_j - z_k)|,$$

where, as always, $y_j - y = O(h)$ and $y_k - y' = O(h)$. Now, since the boxes $\bar{B}_j$ and $\bar{B}_k$ are separated by $O(h)$, we have the bound

$$\int |R_2(y, y')|\, dy' \leq Ch \sum_{k \neq j} \int_{\bar{B}_k} |\nabla^2 K(z_j - z_k)|\, dy'$$

$$\leq Ch \int_{|\xi - \xi'| \geq \alpha(t)h} \frac{1}{|\xi - \xi'|^3}\, d\xi' \leq C(t).$$

This argument is symmetric in y and y' and so it also gives the other bond needed for Young's inequality,

$$\int |R_2(y, y')|\, dy \leq C(t).$$

This completes the proof of the lemma and of the theorem.

A numerical experiment also demonstrates the second-order convergence of the point vortex method. We started with initial conditions (see [12])

$$\omega^0(x) = \begin{cases} \big(1 - |x|^2\big)^7, & 0 \leq |x| \leq 1, \\ 0, & |x| > 1, \end{cases}$$

and solved the equations (6) using fourth order Runge Kutta (see e.g. [7], p. 346) with $h = \Delta t$. It is easy to compute the exact velocity field and Lagrangian coordinate. The velocity field does not change with time. Particles near the origin complete one rotation at time $t = 4\pi$, while the particles on $|x| = 1$ complete one rotation at $t = 32\pi$. The velocity error is $E(h, t) = \|v - u\|$. Figures 1 and 2 show the velocity error in maximum and l_2 norms, respectively, for $h = 0.2, 0.1, 0.05$. The order of accuracy is the power, r, in $E(h, t) \sim C(t)h^r$. In Figure 3, we plot $r(h, t) = \log_2(E(2h, t)/E(h, t))$, which is an estimate of r. For sufficiently short times, the second-order accuracy ($r = 2$) is borne out. In Figure 4 we compare $E(h, t)$ to $\|\rho(t)\|$(see (11)). The agreement of these two indicates that the method is stable.
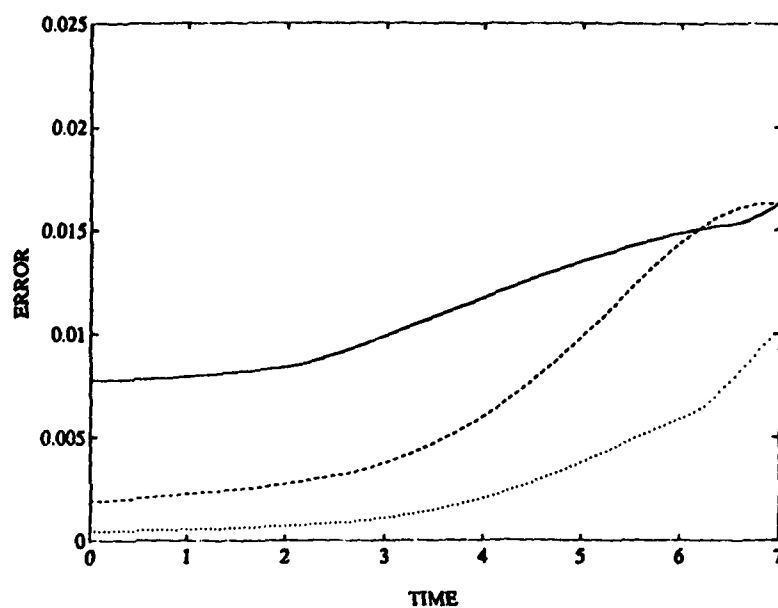
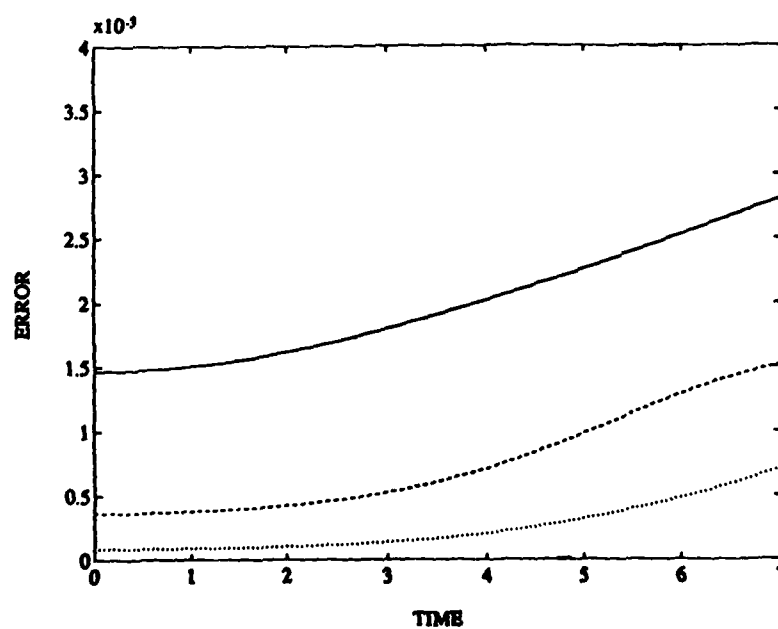Figure 1.   Maximum errors in velocity. h : ($-$)0.2, ($--$)0.1, ($\ldots$)0.05.

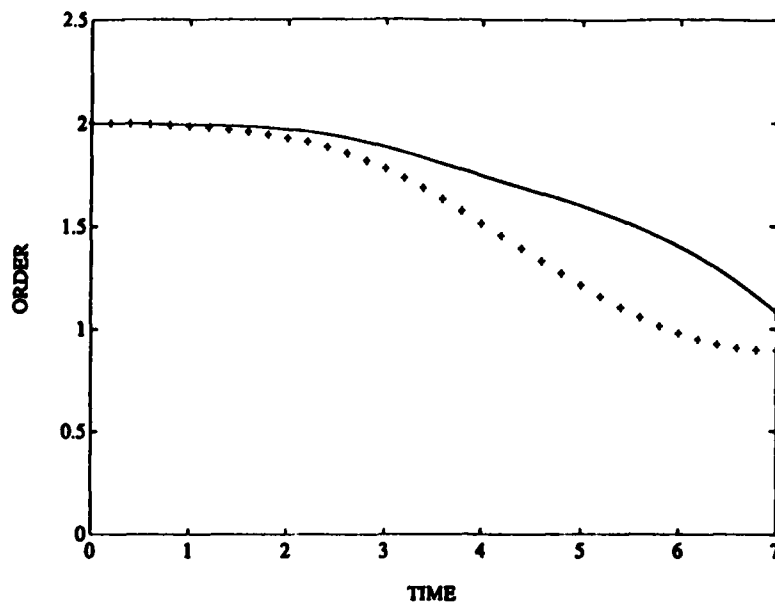Figure 2. l_2 errors in velocity. h : ($-$)0.2, ($--$)0.1, ($\ldots$)0.05.

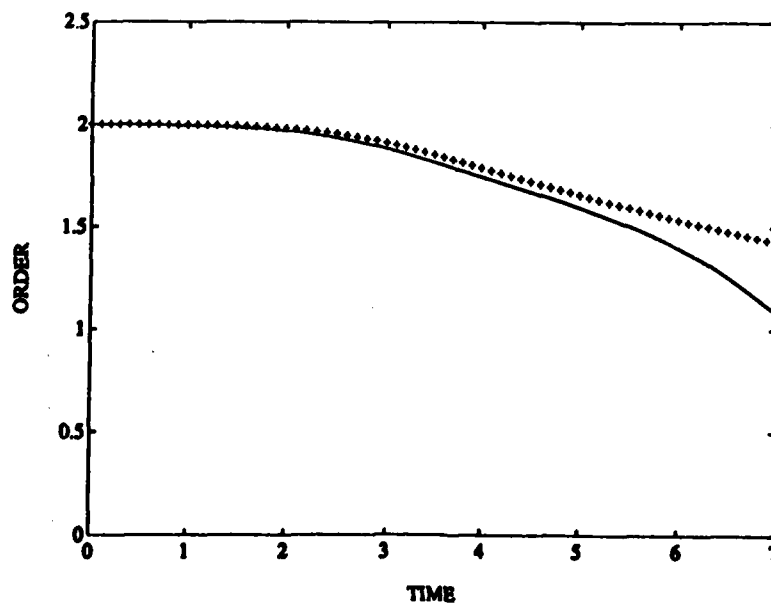Figure 3.   Order of convergence in l_2 norm in velocity. h : (−)0.05 vers 0.1, (+)0.1 vers 0.2.

Figure 4.   Order of convergence in l_2 norm in velocity (−) vers order of convergence in residual (+).

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